Grain Settlement and Fluid Flow Cause the Earthquake Liquefaction of Sand (Supplement)

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Abstract

In the interest of clarity, we provide intermediate representations of our equation so that the derivation is more easily followed.

Details of Derivation

Dimensional Equation

Intermediate steps in the derivation of our equations include many terms that are ultimately negligible. However, rather than attempting to simplify the equations early on, we choose to carry these terms through a comprehensive analysis which prevents over-simplification and makes the assumptions more explicit. The "undrained" assumption mentioned in the paper is an example of a typical simplifying assumption which has in fact led to over-simplification and misled most researchers up to this point.

As mentioned in the paper, the starting point for the dimensional equation is:

$$\frac{d}{dt}\left(\rho\phi A dz\right)dt = -\frac{\partial}{\partial z}\left(A\rho\left(-\frac{k_d}{\mu}\frac{\partial P}{\partial z} + v_g\right)dt\right)dz - \Delta M_{\mathrm{static}} \tag{1}$$

With $\Delta M_{\mathrm{static}}$ provided by setting the left hand side of this equation to zero, and giving the right hand side terms the values they would have under static conditions and solving for $\Delta M_{\mathrm{static}}$:

$$\Delta M_{\mathrm{static}} = \frac{\rho^2 g A}{\mu}\left(\frac{g k_d \rho_0}{K_B} - \frac{k_d}{\rho}\frac{d\rho}{dz} - \frac{\partial k_d}{\partial z}\right)dtdz \tag{2}$$

In addition, the Taylor series equation of state for small deviations in ρ and T can be used.

$$P(\rho, T) = P_0 + \frac{K_B}{\rho_0}(\rho - \rho_0) + K_B \alpha (T - T_0) \tag{3}$$

We solve this equation for ρ and substitute into equation (1).

We then use the comprehensive chain rule application to expand the equation in terms of all the relevant derivatives. In dimensional form, this equation is too large to typeset, however after conversion to dimensionless form we can reduce the size while retaining all terms.

Dimensionless Equation

After substituting the expressions in equations (7) we solved for the dimensionless $\frac{\partial P'}{\partial t'}$ and replaced the dimensionless groups in (10-14) as well as an additional group

$$\gamma_{\text{thermC}} = \frac{K_T \mu \phi_0}{C_v k_{d0} K_B \rho_0} \approx 5 \times 10^{-10}$$

which is a thermal conductivity related dimensionless group (K_T is the thermal conductivity of water) that is clearly negligible and which will ultimately drop out of the analysis. So far this equation is full, retaining all terms from equation (1) in dimensionless form.

$$
\begin{aligned}
\frac{dP'}{dt'} = {} & \frac{\frac{\partial S'}{\partial t'}\, T'\, \gamma_S^2\, \gamma_{\mathrm{vol}}}{\alpha\, C'_v\left(\hat{\phi}\right)\, T_0} \\
& + \gamma_S\, \gamma_{\mathrm{vol}}^2 \left(-\frac{k'_d\, \phi_0\, \frac{\partial P'}{\partial z'}\, \left(P'\, \frac{\partial T'}{\partial z'}\, \gamma_{\mu_T} + T'\right)}{\varphi} - \frac{6\, \left(\phi_0 - 1\right)^2\, \phi_0^2\, \frac{\partial \phi'}{\partial z'}\, T'\, v'_g}{\left(\varphi - 1\right)\, \varphi\, \left(\phi_0 - 2\right)^2} \right. \\
& \qquad \left. - \frac{k'_d\, \phi_0\, \left(P'\, \frac{\partial T'}{\partial z'} - 2\, T'\right)}{\varphi} - \frac{2\, \frac{\partial k'_d}{\partial z'}\, \phi_0\, P'\, T'}{\varphi}\right) \\
& + \gamma_S\, \gamma_{\mathrm{vol}} \left(-\frac{k'_d\, \phi_0\, \frac{\partial P'}{\partial z'}\, \frac{\partial T'}{\partial z'}\, \left(\gamma_{\mu_T} - \varphi + 1\right)}{\varphi} + \frac{6\, \left(\phi_0 - 1\right)\, \phi_0\, \frac{\partial T'}{\partial z'}\, v'_g}{\varphi\, \left(\phi_0 - 2\right)} - \frac{k'_d\, \phi_0\, \frac{\partial T'}{\partial z'}}{\varphi} \right. \\
& \qquad \left. - \frac{k'_d\, \phi_0\, \frac{\partial^2 P'}{\partial z'^2}\, T'}{\varphi} - \frac{\frac{\partial k'_d}{\partial z'}\, \phi_0\, \frac{\partial P'}{\partial z'}\, T'}{\varphi} + \frac{\left(\varphi - 2\right)\, \left(\phi_0 - 1\right)\, \phi_0\, \frac{\partial \phi'}{\partial t'}\, T'}{\left(\varphi - 1\right)\, \varphi\, \left(\phi_0 - 2\right)} - \frac{2\, \frac{\partial k'_d}{\partial z'}\, \phi_0\, T'}{\varphi}\right) \\
& + \gamma_S^2\, \gamma_{\mathrm{vol}}^2 \left(\frac{k'_d\, \phi_0\, \frac{\partial P'}{\partial z'}\, T'\, \frac{\partial T'}{\partial z'}\, \gamma_{\mu_T}}{\varphi} + \frac{k'_d\, \phi_0\, T'\, \frac{\partial T'}{\partial z'}}{\varphi} + \frac{\frac{\partial k'_d}{\partial z'}\, \phi_0\, {T'}^2}{\varphi}\right) \\
& - \frac{k'_d\, \phi_0\, {T'}^2\, \gamma_S^2\, \gamma_{\mathrm{vol}}^3}{\varphi} \\
& + \frac{2\, k'_d\, \phi_0\, P'\, T'\, \gamma_S\, \gamma_{\mathrm{vol}}^3}{\varphi} \\
& - \frac{k'_d\, \phi_0\, {P'}^2\, \gamma_{\mathrm{vol}}^3}{\varphi} \\
& + \left(\frac{6\, \left(\phi_0 - 1\right)^2\, \phi_0^2\, \frac{\partial \phi'}{\partial z'}\, P'\, v'_g}{\left(\varphi - 1\right)\, \varphi\, \left(\phi_0 - 2\right)^2} + \frac{k'_d\, \phi_0\, P'\, \frac{\partial P'}{\partial z'}}{\varphi} + \frac{\frac{\partial k'_d}{\partial z'}\, \phi_0\, {P'}^2}{\varphi} - \frac{2\, k'_d\, \phi_0\, P'}{\varphi}\right) \gamma_{\mathrm{vol}}^2 \\
& + \left(-\frac{6\, \left(\phi_0 - 1\right)\, \phi_0\, \frac{\partial P'}{\partial z'}\, v'_g}{\varphi\, \left(\phi_0 - 2\right)} + \frac{6\, \left(\phi_0 - 1\right)^2\, \phi_0^2\, \frac{\partial \phi'}{\partial z'}\, v'_g}{\left(\varphi - 1\right)\, \varphi\, \left(\phi_0 - 2\right)^2} + \frac{k'_d\, \phi_0\, P'\, \frac{\partial^2 P'}{\partial z'^2}}{\varphi} \right. \\
& \qquad + \frac{k'_d\, \phi_0\, \left(\frac{\partial P'}{\partial z'}\right)^2}{\varphi} + \frac{\frac{\partial k'_d}{\partial z'}\, \phi_0\, P'\, \frac{\partial P'}{\partial z'}}{\varphi} + \frac{k'_d\, \phi_0\, \frac{\partial P'}{\partial z'}}{\varphi} - \frac{\left(\varphi - 2\right)\, \left(\phi_0 - 1\right)\, \phi_0\, \frac{\partial \phi'}{\partial t'}\, P'}{\left(\varphi - 1\right)\, \varphi\, \left(\phi_0 - 2\right)} \\
& \qquad \left. + \frac{2\, \frac{\partial k'_d}{\partial z'}\, \phi_0\, P'}{\varphi} - \frac{k'_d\, \phi_0}{\varphi}\right) \gamma_{\mathrm{vol}} \\
& - \frac{\gamma_{\mathrm{thermC}}\, \frac{\partial^2 T'}{\partial z'^2}\, \gamma_S}{\left(G_{\mathrm{gr}}\, \varphi - \varphi - G_{\mathrm{gr}}\right)\, C'_v\left(\hat{\phi}\right)} \\
& + \frac{\frac{\partial S'}{\partial t'}\, \gamma_S}{C'_v\left(\hat{\phi}\right)} \\
& + \frac{k'_d\, \phi_0\, \frac{\partial^2 P'}{\partial z'^2}}{\varphi} \\
& + \frac{\frac{\partial k'_d}{\partial z'}\, \phi_0\, \frac{\partial P'}{\partial z'}}{\varphi} \\
& - \frac{\left(\varphi - 2\right)\, \left(\phi_0 - 1\right)\, \phi_0\, \frac{\partial \phi'}{\partial t'}}{\left(\varphi - 1\right)\, \varphi\, \left(\phi_0 - 2\right)} \\
& + \frac{\frac{\partial k'_d}{\partial z'}\, \phi_0}{\varphi}
\end{aligned} \tag{4}
$$

Taking this full equation we can simplify easily using only the first order terms from the Taylor series about zero for the small variables $\gamma_{\mathrm{vol}}, \gamma_S, \gamma_{\mathrm{thermC}}$ leading to:

$$
\begin{aligned}
\frac{dP'}{dt'} = \Bigg(& -\frac{6\,(\phi_0-1)\,\phi_0\,\frac{\partial P'}{\partial z'}\,v_g'}{\varphi\,(\phi_0-2)} + \frac{6\,(\phi_0-1)^2\,\phi_0^2\,\frac{\partial \phi'}{\partial z'}\,v_g'}{(\varphi-1)\,\varphi\,(\phi_0-2)^2} + \frac{k_d'\,\phi_0\,P'\,\frac{\partial^2 P'}{\partial z'^2}}{\varphi} \\
& + \frac{k_d'\,\phi_0\,\left(\frac{\partial P'}{\partial z'}\right)^2}{\varphi} + \frac{\frac{\partial k_d'}{\partial z'}\,\phi_0\,P'\,\frac{\partial P'}{\partial z'}}{\varphi} + \frac{k_d'\,\phi_0\,\frac{\partial P'}{\partial z'}}{\varphi} \\
& - \frac{(\varphi-2)\,(\phi_0-1)\,\phi_0\,\frac{\partial \phi'}{\partial t'}\,P'}{(\varphi-1)\,\varphi\,(\phi_0-2)} + \frac{2\,\frac{\partial k_d'}{\partial z'}\,\phi_0\,P'}{\varphi} - \frac{k_d'\,\phi_0}{\varphi} \Bigg)\,\gamma_{\mathrm{vol}} \\
& + \frac{\frac{\partial S'}{\partial t'}\,\gamma_S}{C_v'\left(\hat{\phi}\right)} \\
& + \frac{k_d'\,\phi_0\,\frac{\partial^2 P'}{\partial z'^2}}{\varphi} \\
& + \frac{\frac{\partial k_d'}{\partial z'}\,\phi_0\,\frac{\partial P'}{\partial z'}}{\varphi} \\
& - \frac{(\varphi-2)\,(\phi_0-1)\,\phi_0\,\frac{\partial \phi'}{\partial t'}}{(\varphi-1)\,\varphi\,(\phi_0-2)} \\
& + \frac{\frac{\partial k_d'}{\partial z'}\,\phi_0}{\varphi}
\end{aligned}
\tag{5}
$$

We next notice that $\gamma_{\mu T}$ which is related to the rate of change of viscosity is large (≈ -118), and that the combination $\gamma_{\mathrm{vol}}\gamma_S\gamma_{\mu T}$ was eliminated by the previous Taylor series as second order, even though this combination is not necessarily second order in size due to the presence of the large viscosity related coefficient. We restore terms multiplied by this combination of coefficients.

$$
\begin{aligned}
\frac{dP'}{dt'} = &-\frac{k'_d\,\phi_0\,\frac{\partial P'}{\partial z'}\,\frac{\partial T'}{\partial z'}\,\gamma_S\,\gamma_{\mathrm{vol}}\,\gamma_{\mu_T}}{\varphi} \\
&+\Bigg(-\frac{6\,(\phi_0-1)\,\phi_0\,\frac{\partial P'}{\partial z'}\,v'_g}{\varphi\,(\phi_0-2)}+\frac{6\,(\phi_0-1)^2\,\phi_0^2\,\frac{\partial \phi'}{\partial z'}\,v'_g}{(\varphi-1)\,\varphi\,(\phi_0-2)^2}+\frac{k'_d\,\phi_0\,P'\,\frac{\partial^2 P'}{\partial z'^2}}{\varphi} \\
&\qquad+\frac{k'_d\,\phi_0\,\left(\frac{\partial P'}{\partial z'}\right)^2}{\varphi}+\frac{\frac{\partial k'_d}{\partial z'}\,\phi_0\,P'\,\frac{\partial P'}{\partial z'}}{\varphi}+\frac{k'_d\,\phi_0\,\frac{\partial P'}{\partial z'}}{\varphi} \\
&\qquad-\frac{(\varphi-2)\,(\phi_0-1)\,\phi_0\,\frac{\partial \phi'}{\partial t'}\,P'}{(\varphi-1)\,\varphi\,(\phi_0-2)}+\frac{2\,\frac{\partial k'_d}{\partial z'}\,\phi_0\,P'}{\varphi}-\frac{k'_d\,\phi_0}{\varphi}\Bigg)\,\gamma_{\mathrm{vol}} \\
&+\frac{\frac{\partial S'}{\partial t'}\,\gamma_S}{C'_v\left(\hat{\phi}\right)} \\
&+\frac{k'_d\,\phi_0\,\frac{\partial^2 P'}{\partial z'^2}}{\varphi} \\
&+\frac{\frac{\partial k'_d}{\partial z'}\,\phi_0\,\frac{\partial P'}{\partial z'}}{\varphi} \\
&-\frac{(\varphi-2)\,(\phi_0-1)\,\phi_0\,\frac{\partial \phi'}{\partial t'}}{(\varphi-1)\,\varphi\,(\phi_0-2)} \\
&+\frac{\frac{\partial k'_d}{\partial z'}\,\phi_0}{\varphi}
\end{aligned}
\tag{6}
$$

The final stages are an asymptotic analysis of the perturbation terms to retain only those terms which are leading order. In order for the perturbations to be important, we must have the unperturbed equation (15) from the paper in near-equilibrium (otherwise it will dominate). Since all the variables themselves are $O(1)$, in order for the perturbation terms to be appreciable, we need one of the derivatives to be large. For example, if $\frac{\partial kd'}{\partial z'}$ is large we can zoom into this small spatial region by rescaling $z' \to \epsilon\hat{z}$. The dominant terms with the right order of magnitude to balance will have a factor of $1/\epsilon^2$ which includes the $\frac{\partial^2 P'}{\partial z'^2}$, $(\frac{\partial P'}{\partial z'})^2$, and $\frac{\partial k'_d}{\partial z'}\frac{\partial P'}{\partial z'}$ terms, as well as the previously mentioned term involving $\gamma_{\mu T}$, hence they are retained. All remaining perturbation terms are dropped, and this, together with some organizational substitutions leads to our equation (14).

Liquefiable Inter-Layer Example

In the interest of brevity of the main article the graphs for the liquefiable inter-layer example were left out of the main paper. They are reproduced here:

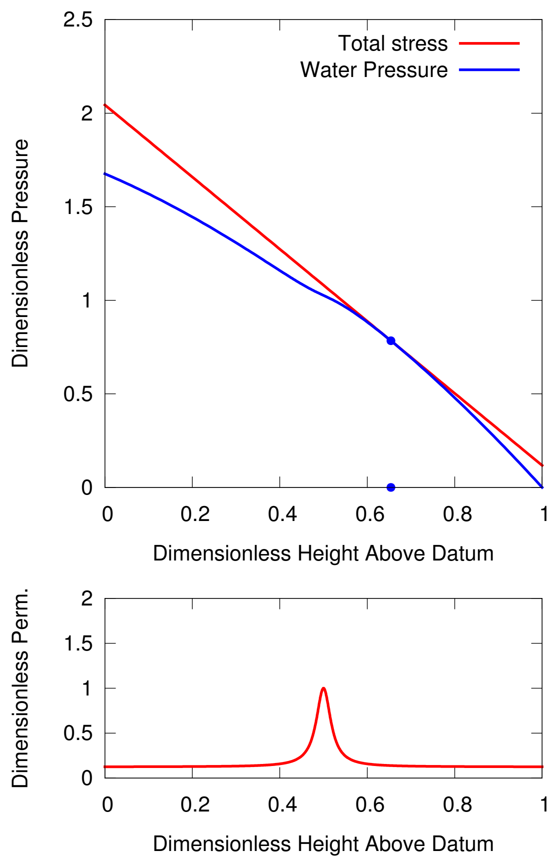

Figure 1: A high permeability layer acts as a drain for water pressure in the lower region and a source for water pressure in the upper region. Liquefaction occurs above the high permeability layer. This result occurs whether we use uniform forcing throughout or localized forcing only within the high permeability layer.